Chapter 09 Division of Polynomials

Handout – 01
Definitions & Formulas

Learning Outcomes

After studying this handout, you will be able to:

- Understand the meaning of a polynomial and its degree.
- Perform the division of polynomials.
- Identify the quotient and remainder.
- Use the Remainder Theorem and Factor Theorem.
- Find the value of a polynomial at a given number.

Definition: Polynomial

A **polynomial in x** is an algebraic expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants, $a_n \neq 0$ and n is a non-negative integer.

Degree of a Polynomial

- The degree of a polynomial is the highest power of x with non-zero coefficient.

Example: If $P(x) = 3x^4 - 2x^2 + 5$,
$\deg P(x) = 4$.

Division Algorithm

If $P(x)$ and $D(x)$ $(D(x) \neq 0)$ are polynomials, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

where $\deg R(x) < \deg D(x)$.

- $Q(x)$: Quotient
- $R(x)$: Remainder

Types of Division

1 Exact Division

If $R(x) = 0$, then $D(x)$ is a factor of $P(x)$.

2 Inexact Division

If $R(x) \neq 0$, then $D(x)$ is not a factor of $P(x)$.

Remainder Theorem

If a polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.

Example: If $P(x) = 2x^3 - 5x + 1$, then the remainder on division by $(x - 2)$ is

$P(2) = 2(2)^3 - 5(2) + 1 = 16 - 10 + 1 =$ **7.**

Factor Theorem

A polynomial $P(x)$ is divisible by $(x - a)$ if and only if $P(a) = 0$.

Example: If $P(x) = x^3 - 3x^2 - 4x + 12$, then

$P(2) = 8 - 12 - 8 + 12 = 0$

$\therefore$ $(x - 2)$ is a factor of $P(x)$.

Important Formulas / Results

1	**Division Algorithm:**	$P(x) = D(x)Q(x) + R(x)$
2	**Remainder:**	$R = P(a)$ (when divisor $= x - a$)
3	**Factor Test:**	$(x - a) \mid P(x) \Leftrightarrow P(a) = 0$
4	**Degree of Remainder:**	$\deg R(x) < \deg D(x)$

Key Remarks

- A polynomial of degree n has $n + 1$ terms (at most).
- The remainder is either zero or a polynomial of degree less than the degree of the divisor.
- If $(x - a)$ and $(x - b)$ are factors, then $(x - a)(x - b)$ is also a factor.

Quick Example

Find the remainder when $P(x) = x^3 + 4x^2 - x + 6$ is divided by $(x + 2)$.

Solution: $R = P(-2) = (-2)^3 + 4(-2)^2 - (-2) + 6$

$= -8 + 16 + 2 + 6 =$ **16.**

Exercise 9.1

Q1. Find remainder and quotient by simplifying the following :

(i) $(3x^2-x+2) \div (x-1)$

Sol:

$$
\begin{array}{r|l}
x-1 & 3x+2 \\
\hline
 & 3x^2-x+2 \\
 & -(3x^2-3x) \\
\hline
 & 2x+2 \\
 & -(2x-2) \\
\hline
 & 4
\end{array}
$$

Quotient = $3x+2$, Remainder = 4.

(ii) $(x^3+12x^2-3x+4) \div (x-2)$

Sol:

$$
\begin{array}{r|l}
x-2 & x^2+14x+25 \\
\hline
 & x^3+12x^2-3x+4 \\
 & -(x^3-2x^2) \\
\hline
 & 14x^2-3x+4 \\
 & -(14x^2-28x) \\
\hline
 & 25x+4 \\
 & -(25x-50) \\
\hline
 & 54
\end{array}
$$

Quotient = $x^2+14x+25$, Remainder = 54.

(iii) $(x^4-5x^3-8x^2+13x+12) \div (x-6)$

Sol:

$$
\begin{array}{r|l}
x-6 & x^3+x^2-2x+1 \\
\hline
 & x^4-5x^3-8x^2+13x+12 \\
 & -(x^4-6x^3) \\
\hline
 & x^3-8x^2+13x+12 \\
 & -(x^3-6x^2) \\
\hline
 & -2x^2+13x+12 \\
 & -(-2x^2+12x) \\
\hline
 & x+12 \\
 & -(x-6) \\
\hline
 & 18
\end{array}
$$

Quotient = x^3+x^2-2x+1 , Remainder = 18.

(iv) $(5x^4-3x^3+2x^2-1) \div (x^2+4)$

Sol:

$$
\begin{array}{r|l}
x^2+4 & 5x^2-3x-18 \\
\hline
 & 5x^4-3x^3+2x^2+0x-1 \\
 & -(5x^4+20x^2) \\
\hline
 & -3x^3-18x^2+0x-1 \\
 & -(-3x^3-12x) \\
\hline
 & -18x^2+12x-1 \\
 & -(-18x^2-72) \\
\hline
 & 12x+71
\end{array}
$$

Quotient = $5x^2-3x-18$, Remainder = $12x+71$.

(v) $(3x^4-5x^3+4x-6) \div (x^2-3x+5)$

Sol:

$$
\begin{array}{r|l}
x^2-3x+5 & 3x^2+4x+27 \\
\hline
 & 3x^4-5x^3+0x^2+4x-6 \\
 & -(3x^4-9x^3+15x^2) \\
\hline
 & 4x^3-15x^2+4x-6 \\
 & -(4x^3-12x^2+20x) \\
\hline
 & -3x^2-16x-6 \\
 & -(-3x^2+9x-15) \\
\hline
 & -25x+9
\end{array}
$$

Quotient = $3x^2+4x+27$,
Remainder = $-25x+9$.

Exercise 9.1

Q2. Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial.

Remainder Theorem — If a polynomial $P(x)$ is divided by $(x-a)$, then the remainder is $R = P(a)$.

If the divisor is $(x+a)$, then use $x = -a$.

(i) $P(x) = x^2 + 5x + 6$, $x - 2$

Sol: Here, $a = 2$

$$R = P(2) = (2)^2 + 5(2) + 6$$
$$= 4 + 10 + 6$$
$$= 20$$

Remainder = 20

(ii) $P(x) = x^3 + 5x^2 + 6$, $x + 1$

Sol: Here, $a = -1$

$$R = P(-1) = (-1)^3 + 5(-1)^2 + 6$$
$$= -1 + 5 + 6$$
$$= 10$$

Remainder = 10

(iii) $P(x) = x^4 + x^3 + x^2 + x + 1$, $x - 1$

Sol: Here, $a = 1$

$$R = P(1) = (1)^4 + (1)^3 + (1)^2 + (1) + 1$$
$$= 1 + 1 + 1 + 1 + 1$$
$$= 5$$

Remainder = 5

(iv) $P(x) = x^4 + x^2 + 1$, $x + 3$

Sol: Here, $x + 3 = 0 \Rightarrow x = -3$

$$R = P(-3) = (-3)^4 + (-3)^2 + 1$$
$$= 81 + 9 + 1$$
$$= 91$$

Remainder = 91

(v) $P(x) = x^4 + x^3 + 2$, $x + 2$

Sol: Here, $x + 2 = 0 \Rightarrow x = -2$

$$R = P(-2) = (-2)^4 + (-2)^3 + 2$$
$$= 16 - 8 + 2$$
$$= 10$$

Remainder = 10

Final Answers

Part	Remainder
(i)	20
(ii)	10
(iii)	5
(iv)	91
(v)	10

Exercise 9.1

Q3. Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

Factor Theorem If $P(x)$ is a polynomial and $(x-a)$ is a factor of $P(x)$, then $P(a)=0$.

(i) $x+1$, x^3-1

Sol: Here, $x+1=0 \Rightarrow x=-1$

Let $P(x)=x^3-1$.

$P(-1)=(-1)^3-1$

$=-1-1$

$=-2\neq 0$

$\therefore$ $x+1$ is not a factor of x^3-1.

(ii) $x-2$, x^2-5x+6

Sol: Here, $x-2=0 \Rightarrow x=2$

Let $P(x)=x^2-5x+6$.

$P(2)=(2)^2-5(2)+6$

$=4-10+6$

$=0$

$\therefore$ $x-2$ is a factor of x^2-5x+6.

(iii) $x+1$, x^3+x^2+x-3

Sol: Here, $x+1=0 \Rightarrow x=-1$

Let $P(x)=x^3+x^2+x-3$.

$P(-1)=(-1)^3+(-1)^2+(-1)-3$

$=-1+1-1-3$

$=-4\neq 0$

$\therefore$ $x+1$ is not a factor of x^3+x^2+x-3.

(iv) $x-2$, x^3+x^2-7x+2

Sol: Here, $x-2=0 \Rightarrow x=2$

Let $P(x)=x^3+x^2-7x+2$.

$P(2)=(2)^3+(2)^2-7(2)+2$

$=8+4-14+2$

$=0$

$\therefore$ $x-2$ is a factor of x^3+x^2-7x+2.

(v) $x-3$, $x^4-3x^2+x^2-x+1$

Sol: Here, $x-3=0 \Rightarrow x=3$

Let $P(x)=x^4-3x^2+x^2-x+1$

$=x^4-2x^2-x+1$.

$P(3)=3^4-2(3)^2-3+1$

$=81-18-3+1$

$=61\neq 0$

$\therefore$ $x-3$ is not a factor of $x^4-3x^2+x^2-x+1$.

Final Answers

Part	Conclusion
(i)	Not a factor
(ii)	Is a factor
(iii)	Not a factor
(iv)	Is a factor
(v)	Not a factor

Exercise 9.1

Q4. Use synthetic division to show that x is the zero of the polynomial and use the result to factorize the polynomial completely.

Synthetic Division — If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$, Synthetic division gives the quotient and the remaining factors.

(i) $P(x) = x^3 - 7x + 6$, $x = 2$

Sol: Let $P(x) = x^3 - 7x + 6$.

Using synthetic division with $x = 2$:

2	1	0	−7	6
		2	4	−6
	1	2	−3	0

Remainder $= 0 \Rightarrow x = 2$ is a zero.

Quotient $= x^2 + 2x - 3$
$= (x + 3)(x - 1)$

$\therefore\ x^3 - 7x + 6 = (x - 2)(x + 3)(x - 1).$

(ii) $P(x) = x^3 - 28x - 48$, $x = -4$

Sol: Let $P(x) = x^3 - 28x - 48$.

Using synthetic division with $x = -4$:

−4	1	0	−28	−48
		−4	16	48
	1	−4	−12	0

Remainder $= 0 \Rightarrow x = -4$ is a zero.

Quotient $= x^2 - 4x - 12$
$= (x - 6)(x + 2)$

$\therefore\ x^3 - 28x - 48 = (x + 4)(x - 6)(x + 2).$

(iiii) $P(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, $x = 2$, $x = -3$

Sol: Let $P(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$.

Step 1: Using synthetic division with $x = 2$:

2	2	7	−4	−27	−18
		4	22	36	18
	2	11	18	9	0

Remainder $= 0$
$\Rightarrow x = 2$ is a zero.
Quotient $= 2x^3 + 11x^2 + 18x + 9.$

Step 2: Using synthetic division again with $x = -3$:

−3	2	11	18	9
		−6	−15	−9
	2	5	3	0

Remainder $= 0$
$\Rightarrow x = -3$ is also a zero.
Quotient $= 2x^2 + 5x + 3$
$= (2x + 3)(x + 1).$

Hence, $2x^4 + 7x^3 - 4x^2 - 27x - 18 = (x - 2)(x + 3)(2x + 3)(x + 1).$

Final Answers

Part	Factorization
(i)	$x^3 - 7x + 6 = (x - 2)(x + 3)(x - 1)$
(ii)	$x^3 - 28x - 48 = (x + 4)(x - 6)(x + 2)$
(iiii)	$2x^4 + 7x^3 - 4x^2 - 27x - 18 = (x - 2)(x + 3)(2x + 3)(x + 1)$

Exercise 9.1

Q5. Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Sol: Let $P(x) = x^4 - 10x^2 - 2x + 4$.
Since the divisor is $x + 3$, take $x = -3$.
The coefficients of $P(x)$ are $1, 0, -10, -2, 4$.

-3	1	0	-10	-2	4
		-3	9	3	-3
	1	-3	-1	1	1

$\therefore$ Quotient $= x^3 - 3x^2 - x + 1$
Remainder $= 1$.

Hence, $x^4 - 10x^2 - 2x + 4 = (x + 3)(x^3 - 3x^2 - x + 1) + 1$.

Q6. If $x + 1$ and $x - 2$ are factors of $x^3 - px^2 + qx + 2$. Using synthetic division, find the values of p and q.

Sol: Let $P(x) = x^3 - px^2 + qx + 2$.

Since $x + 1$ is a factor, take $x = -1$.

-1	1	$-p$	q	2
		-1	p	$-q$
	1	$-(p+1)$	$q+p$	$2-q$

As $x + 1$ is a factor, remainder $= 0$
$\Rightarrow 2 - q = 0 \Rightarrow q = 2$.

Since $x - 2$ is a factor, take $x = 2$.

2	1	$-p$	q	2
		2	4	$-2p + 4$
	1	$2-p$	$q+4-2p$	0

As $x - 2$ is a factor, remainder $= 0$
$\Rightarrow q + 4 - 2p = 0$.
Substitute $q = 2$:
$2 + 4 - 2p = 0 \Rightarrow 6 - 2p = 0 \Rightarrow p = 3$.

Thus, $p = 3$ and $q = 2$.

Q7. When the polynomial $4x^4 + 2x^3 + kx^2 + 13$ is divided by $x + 1$, the remainder is 16. Find the value of k.

Sol: Let $P(x) = 4x^4 + 2x^3 + kx^2 + 0 \cdot x + 13$.
Since the divisor is $x + 1$, take $x = -1$.

-1	4	2	k	0	13
		-4	2	$-k$	$-k$
	4	-2	$k+2$	$-k$	$13+k$

Remainder $= 13 + k$.
Given that remainder $= 16$, so
$13 + k = 16 \Rightarrow k = 3$.

Hence, $k = 3$.

Exercise 9.1

Q5. Use synthetic division to find the quotient and the remainder when the polynomial $x^4 - 10x^2 - 2x + 4$ is divided by $x + 3$.

Sol: Let $P(x) = x^4 - 10x^2 - 2x + 4$.
Divisor $x + 3 \Rightarrow$ use $x = -3$.
Coefficients: 1, 0, −10, −2, 4

−3	1	0	−10	−2	4
		−3	9	3	−3
	1	−3	−1	1	1

Quotient:
$Q(x) = x^3 - 3x^2 - x + 1$

Remainder:
$R = 1$

$\therefore\ x^4 - 10x^2 - 2x + 4 = (x+3)(x^3 - 3x^2 - x + 1) + 1.$

Q6. If $x + 1$ and $x - 2$ are factors of $x^3 - px^2 + qx + 2$. Using synthetic division, find the values of p and q.

Sol: Let $P(x) = x^3 - px^2 + qx + 2$.
Since $x + 1$ is a factor, use $x = -1$.

−1	1	−p	q	2
		p	q	−q
	1	−(p+1)	q−p	2−q

Remainder = 0 $\Rightarrow$ $2 - q = 0 \Rightarrow q = 2$.
Thus, $P(x) = x^3 - px^2 + 2x + 2$.

Since $x - 2$ is also a factor, use $x = 2$.

2	1	−p	2	2
		2	−2p+4	12
	1	2−p	6−2p	14

Remainder = 0 $\Rightarrow$ $6 - 2p = 0$
$\Rightarrow$ $p = 3$.

$\therefore\ p = 3,\quad q = 2.$

Q7. When the polynomial $4x^4 + 2x^3 + kx^2 + 13$ is divided by $x + 1$, the remainder is 16. Find the value of k.

Sol: Let $P(x) = 4x^4 + 2x^3 + kx^2 + 13$.
Divisor $x + 1 \Rightarrow$ use $x = -1$.
Coefficients: 4, 2, k, 0, 13

−1	4	2	k	0	13
		−4	2	−k	−k
	4	−2	k+2	−k	13+k

Remainder = 16 $\Rightarrow$ $13 + k = 16$
$\Rightarrow$ $k = 3$.

$\therefore\ k = 3.$

Final Answers

Q.No.	Result
(5)	$Q(x) = x^3 - 3x^2 - x + 1$, $R = 1$
(6)	$p = 3$, $q = 2$
(7)	$k = 3$

Exercise 9.1

Q9. Use factor theorem to find the values of p and q if $x+1$ and $x-2$ are the factors of the polynomial x^3+px^2+qx+3.

Sol: Let $P(x) = x^3+px^2+qx+3$.

Since $x+1$ is a factor, $P(-1)=0$.

$$\begin{aligned} P(-1) &= (-1)^3+p(-1)^2+q(-1)+3 \\ &= -1+p-q+3 \\ &= p-q+2=0 \\ \Rightarrow\ & p-q=-2 \quad \cdots (1) \end{aligned}$$

Since $x-2$ is also a factor, $P(2)=0$.

$$\begin{aligned} P(2) &= (2)^3+p(2)^2+q(2)+3 \\ &= 8+4p+2q+3 \\ &= 4p+2q+11=0 \\ \Rightarrow\ & 4p+2q=-11 \quad \cdots (2) \end{aligned}$$

Solving (1) and (2):

From (1): $p-q=-2 \Rightarrow q=p+2$

Substitute in (2):

$$\begin{aligned} 4p+2(p+2) &= -11 \\ 4p+2p+4 &= -11 \\ 6p &= -15 \Rightarrow p=-\frac{5}{2} \end{aligned}$$

Then $q = p+2 = -\frac{5}{2}+2 = -\frac{1}{2}$

$$\therefore\ p=-\frac{5}{2},\quad q=-\frac{1}{2}.$$

Q10. Use factor theorem to find the values of a and b if -2 and 2 are the roots of the polynomial $2x^3+4x^2+ax+b$.

Sol: Let $P(x) = 2x^3+4x^2+ax+b$.

Since -2 is a root, $P(-2)=0$.

$$\begin{aligned} P(-2) &= 2(-2)^3+4(-2)^2+a(-2)+b \\ &= -16+16-2a+b \\ &= -2a+b=0 \\ \Rightarrow\ & b=2a \quad \cdots (1) \end{aligned}$$

Since 2 is also a root, $P(2)=0$.

$$\begin{aligned} P(2) &= 2(2)^3+4(2)^2+a(2)+b \\ &= 16+16+2a+b \\ &= 2a+b+32=0 \\ \Rightarrow\ & 2a+b=-32 \quad \cdots (2) \end{aligned}$$

Solving (1) and (2):

Substitute $b=2a$ in (2):

$$\begin{aligned} 2a+2a &= -32 \\ 4a &= -32 \\ \Rightarrow\ a &= -8 \end{aligned}$$

Then $b = 2a = 2(-8) = -16$

$$\therefore\ a=-8,\quad b=-16.$$